Solutions to 《Discrete Mathematics (I)》 Exercises 1 Sets

(Due date: 1st June, 2020 Computer Science and Technology 19(1) )

This is the sample problem sets that is very similar to the final examination problem sets. I will give the detail solutions in the next two weeks and we will also discuss the following problems in the upcoming lessons. Before that, hope you can finish all the problems and send me the answer.

Student ID ___________________ Name ___________________ Score ____________

I. Fill in the blanks (30’, for each blank 2 grades)

1. Let $A$ be a set, if____________________________________, $A$ is finite set. If________________________________________________, then set $A$ is called countable.

Solution: This problem corresponds to the definition of finite set and countable set. See page 121 of your textbook. If $A$ is empty set or set $A$ has the same cardinality of set $\{0,1,2,\ldots,n-1\}$ where $n$ is a nonnegative integer, then set $A$ is finite set. Or you can just follow the definition of your textbook If there are exactly $n$ distinct elements in $A$ where $n$ is a nonnegative integer, $A$ is finite set. Both of them are right. According to the definition in the textbook (see page 171), a set that is either finite or has the same cardinality as the set of positive integers is called countable. Or use the description of functions, you can say, if there exists a bijection between integer set $\mathbb{N}$ and set $A$, then set $A$ is called countable. They are equivalent.

This problem exams the skill: think with discrete mathematics.

2. Let two sets $A$ and $B$, and we have $|A|=n$, $|B|=m$, from $A$ to $B$, there are (how many) ________ different relations, there are (how many) ____________ different functions. If $n=5$, then there are total number of ___________ total order relations. If $n=m=3$, then from $A$ to $B$ can produce the number of ________ different bijections.

Solution: We let set $A$ be $\{a_1,a_2,\ldots,a_n\}$ and set $B$ be $\{b_1,b_2,\ldots,b_m\}$, then according to the definition of relations, $a\mathrel{R}b$, where $a\in A$, $b\in B$, there are $n\times m$ elements in set $A\times B$. According to the definition of relations, the relation is a set. Since there are $2^{nm}$ subsets, then there must be the same number of relations from $A$ to $B$. For functions, according to the definition of functions, each element in pre-image set must has its image and there are $m$ different choices. Therefore, there are $m\times m\times\ldots\times m = m^n$ different functions. Open your textbook to page 619,
the definition of total ordering states that: when every two elements in the sets are comparable, the relation is called a total ordering. The elements a and b of a poset \((S, \preceq)\) are called comparable if either \(a \preceq b\) or \(b \preceq a\). Then it is a chain (see definition of textbook). For the first element of the chain there are 5 choices, for the second, 4 choices, for the third, 3 choices, ..., for the fifth, 1 choice. Thus there are \(5\) different total orderings. To solve the final problem, we have to come back to the definition of injection, surjection and bijection relations. From set \(A\) to set \(B\), for the first element of set \(A\), it has 3 choices to link with one element of set \(B\). The second element of set \(A\), it has 2 choices when the first relation has been determined. Therefore, there are total number of \(3 \times 2 \times 1 = 6\) different bijections.

This problem exams counting numbers, the understanding of functions, and partial orders.

3. The recursive definition of sets is consisted of three parts or three steps, they are:
   (1) ____________________________;
   (2) ____________________________;
   (3) ____________________________;

Solution: This problem is related with the contents on page 349 of your textbook. Step 1, some elements are in the set \(A\) we are now try to define, to demonstrate that set \(A\) is not an empty set. Step 2, use the elements currently in set \(A\) to produce more elements, that is, to construct a method to produce new elements in set \(A\) with existing elements in set \(A\). Step 3, the elements in set \(A\) are obtained using finite steps of step 1 and step 2.

4. Let \(A\) and \(B\) be two sets, then \(A \cap B = B\) if and only if: ____________________________;

   (Fill in the relation between \(A\) and \(B\))
   \(A \oplus B = B\) if and only if (Fill in the blank with relation between set \(A\) and set \(B\))
   ____________________________;

   Solutions: \(A \cap B = B \iff \{x \mid x \in A \land x \in B \iff x = x \in B \iff x \in A \iff B \subseteq A\};\) \(\oplus\) is the symmetric difference operator. \(A \oplus B = A + B - A \cap B\); we can have a guess using Venn diagram. Guess \(A = \emptyset\). Sufficient condition; If \(A = \emptyset\), then \(A \oplus B = \emptyset \cup B = B\)

   Necessary condition; If \(A \neq \emptyset\), then there exists \(x \in A\), there are two sub-cases:
   1) \(x \in B\), then \(x \in A \cap B\), therefore \(x \notin A \oplus B\), so \(A \oplus B \neq B\);
   2) \(x \notin B\), then \(x \in A \oplus B\), we obtain \(A \oplus B \neq B\).
5. Function \( f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \ f((x, y)) = x^2 + y^2 \). Then \( f^{-1}(\{0\}) = \) ______________________.

Solution: \( f^{-1}: \mathbb{N} \to \mathbb{N}, \) if \( x^2 + y^2 = 0, \) then \( x = y = 0. \) Therefore, \( f^{-1}(\{0\}) = (0, 0). \)

6. Function \( f: \mathbb{A} \to \mathbb{B} \) has inverse function if and only if the following conditions are met ______________________.

Solution: According to the definition of function, each image in the set \( \mathbb{B} \) must have pre-image in set \( \mathbb{A}, \) and for inverse function, each element in \( \mathbb{A} \) must have pre-image in \( \mathbb{B}. \) Both directions are one-to-one relations. So there must be bijection.

7. \( \mathbb{A}, \mathbb{B} \) are sets, \( P(\mathbb{A}), P(\mathbb{B}) \) are the power sets, and \( \mathbb{A} \cap \mathbb{B} = \emptyset, \) then \( P(\mathbb{A}) \cap P(\mathbb{B}) = \) ______________________.

Solution:
Solutions: According to the definition of power sets, it is obviously that the answer is \( \{\emptyset\}. \)

8. \( \mathbb{A}, \mathbb{B} \) are two sets, \( \aleph_0 = |\mathbb{B}| < |\mathbb{A}| = \aleph, \) then \( |\mathbb{A} - \mathbb{B}| = \) ______________________.

Solutions: see Chapter 2.5 of your textbook, when an infinite set \( \mathbb{S} \) is countable, we denote the cardinality of \( \mathbb{S} \) by \( \aleph_0 \) (where \( \aleph \) is aleph, the first letter of the Hebrew alphabet). We write \( |\mathbb{S}| = \aleph_0 \) and say that \( \mathbb{S} \) has cardinality “aleph null.” The answer is \( \aleph. \) \( \aleph_0 < \aleph_1 < \aleph_2 < ... \) If we assume that the continuum hypothesis is true, it would follow that \( c = \aleph_1, \) so that \( 2^{\aleph_0} = \aleph_1. \)

II. True or false (24’, each problem worthy 6 marks, state whether it is true or false can earn 3 marks, state the reason why it is true or false earns another 3 marks)

1. \( \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D} \) are any sets; \( f \) is a bijection from set \( \mathbb{A} \) to set \( \mathbb{B}, \) \( g \) is a bijection from set \( \mathbb{C} \) to set \( \mathbb{D}, \) among which for any \( (a, c) \in \mathbb{A} \times \mathbb{C}, \) \( h((a, c)) = (f(a), g(c)) \) holds. Then \( h \) is also bijection.

(1) Please indicate \( \checkmark \) or \( \times \) ( )

(2) Proof:

Solution: (1) Yes, it is true.

(2) Proof: ① \( h \) is a surjection. For any \( (b, d) \in \mathbb{B} \times \mathbb{D}, \) \( b \in \mathbb{B}, \) \( d \in \mathbb{D}. \) Because \( f \) is a bijection from \( \mathbb{A} \) to \( \mathbb{B}, \) \( g \) is a bijection from \( \mathbb{C} \) to \( \mathbb{D}, \) there exists \( a \in \mathbb{A}, \) \( c \in \mathbb{C} \) such that \( f(a) = b, \) \( g(c) = d; \) That is, there exists \( (a, c) \in \mathbb{A} \times \mathbb{C}, \) such that \( h((a, c)) = (f(a), g(c)) = (b, d) \) holds. Therefore, \( h \) is a surjection. ② \( h \) is an injection. For any \( (a_1, c_1) \in \mathbb{A} \times \mathbb{C}, \) \( (a_2, c_2) \in \mathbb{A} \times \mathbb{C}, \) if \( h((a_1, c_1)) = h((a_2, c_2)) \), then \( (f(a_1), g(c_1)) = (f(a_2), g(c_2)) \). Therefore, \( f(a_1) = f(a_2), \) \( g(c_1) = g(c_2). \) Because \( f \) is a bijection
from A to B, g is a bijection from C to D, so that \(a_1 = a_2, \ c_1 = c_2\). Therefore \((a_1, c_1) = (a_2, c_2)\), h is an injection.

2. Let \(R\) be the binary relation on Set \(A\), then \(t(s(R)) = s(t(R))\), \(t(.)\) is the transitive closure, \(s(.)\) is the symmetric closure.
   (1) Please indicate √ or × ( )
   (2) Proof:
   **Solution:** (1) false.
   (2) This problem corresponds to the exercise on page 636 of the textbook. You can give an example to show it is false.

**DEFINITIONS**

A relation \(R\) on a set \(A\) is **transitive** if \((a, b) \in R\) and \((b, c) \in R\) implies \((a, c) \in R\).

The **symmetric closure** of \(R\) is the union of the relation \(R\) with its inverse relation \(R^{-1}\).

**SOLUTION**

(a) For example:

\[A = \{1, 2, 3\}\]
\[R = \{(1, 2), (1, 3)\}\]

Let us first determine the systematic closure:

\[R \cup R^{-1} = \{(1, 2), (1, 3)\} \cup \{(2, 1), (3, 1)\}\]
\[= \{(1, 2), (1, 3), (2, 1), (3, 1)\}\]

Let us next determine the transitive closure of the systematic closure of \(R\):

\[\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2), (3, 1), (3, 3)\}\]

Note that the transitive closure of \(R\) is the relation \(R\) itself (as \(R\) is transitive).

The **systematic closure** of the transitive closure of \(R\) is then:

\[\{(1, 2), (1, 3), (2, 1), (3, 1)\}\]

Note: \(\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2), (3, 1), (3, 3)\} \neq \{(1, 2), (1, 3), (2, 1), (3, 1)\}\)

Thus the transitive closure of the systematic closure of a relation is not the same as the systematic closure of the transitive closure of the relation.

3. Let \(A, B\) be sets, if there exists a surjection from \(A\) to \(B\), then \(|B| \leq |A|\).
   (1) Please indicate √ or × ( )
   (2) Proof:
   **Solution:** (1) It is true.
   (2) According to the definition of surjection, and the definition of function, it is true. Because for any \(b \in B\), there exists \(x \in A\), \(f(a) = b\). for \(f(a1) = b1, f(a2) = b2\), if \(b1 \neq b2\), then \(a1 \neq a2\). Therefore, the conclusion holds.
4. Let $A$ be a set, $R$ is the binary relation on the power set of $A$, $P(A)$, for any $S, T \in P(A)$, $(S, T) \in R$, $R$ is partial ordering relation iff $|S| \leq |T|$.

(1) Please indicate √ or × ( )

(2) Proof:

Solution: (1) It is false.

(2) For the binary relation $R$ on the power set of $A$, $|S| \leq |T|$, $R$ is partial ordering is neither the sufficient condition nor the necessary condition. For example, let $A=\{a, b\}$, $S=\{a\}$, $T=\{b\}$. Because $|S| \leq |T| \text{ and } |T| \leq |S|$, $(S, T) \in R$, and $(T, S) \in R$; Because $S \neq T$, $R$ is not antisymmetric. Therefore, $R$ is not a partial order. We can conclude that $|S| \leq |T|$ is not the sufficient condition. Let $R=\{ (\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\}), (\{a, b\}, \{a\}) \}$, $R$ is the partial order. Because $(\{a, b\}, \{a\}) \in R$, $|S| \leq |T|$ does not hold. Therefore, $|S| \leq |T|$ is not the necessary condition. Based on the above analysis, the conclusion could be made.

III. Integration skills (46’)

1. We have bijection $f: A \rightarrow B$, try to construct another bijection from $P(A)$ to $P(B)$, and prove it is a bijection. (15 scores, give the bijection can earn 5 marks, give the proof can earn 10 marks)

Solution: Construction a function $g: P(A) \rightarrow P(B)$, $\forall x \in P(A)$, let $g(x)=y$, such that $\forall a \in x$, we have $f(a) \in y$, and for any $\forall b \in y$, we have $f^{-1}(b) \in x$.

Prove that $g$ is a bijection from $P(A)$ to $P(B)$.

(1) $g$ is a surjection from $P(A)$ to $P(B)$.

$\forall y \in P(B)$, $\exists x \in P(A)$, for $\forall b \in y$, $f^{-1}(b) \in x$, $x$ is always exists.

(2) $g$ is an injection from $P(A)$ to $P(B)$.

$\forall x_1, x_2 \in P(A)$, if $g(x_1)=g(x_2)=y$, then for $\forall a \in x_1$, $f(a)=g(x_1) \in y$, note that $g(x_2)=y$, therefore $f(a) \in g(x_2)$, according to the definition of function $g$, $a \in x_2$, therefore, $x_1 \subseteq x_2$, similarly, we can prove that $x_2 \subseteq x_1$, therefore, $x_1=x_2$.

2. In a city, there is a hotel, this hotel is very common and nothing different to existing hotels except that the number of rooms is infinite. The room number of the hotel is $1, 2, 3, 4, \ldots$. We can call it the Grand Hotel. The room of the Grand Hotel can be listed as one row $(1, 2, 3, 4, \ldots)$, called countable set.

One day, all rooms were full. There came a business man, insisted to live in the hotel. The boss of this hotel cited the“ Axiom of Hotel”and said, ‘It’s full, full is full, very sorry we cannot provide service for you’. At that time, the clever daughter of the boss came, she said, ‘That’s easy, all guests move from their current room to the next neighboring room’. Then guests originally in room number 1 moved to room number 2, guests originally in room number 2 moved to room number 3···. Finally, room number 1 became vacant, so the late business man can have a good rest there in room number 1.

The next day, the Hilbert hotel came another big business delegation, they said
they had countable infinite representatives must live in the hotel, which had made the
boss’s head hurts. Then the boss’s daughter came again to lib her father from puzzle as
she said, ‘Guests originally in room number 1 move to room number 2, guests
originally in room number 2 move to room number 4⋯⋯, guests originally in room
number k move to room number 2k, therefore, room number 1, room number 3, room
number 5, ⋯⋯those rooms are vacant to hold all the delegation’.

The third day, each representative of this delegation began to play new tricks,
they wanted to let each representative of this delegation to occupy countably infinite
number of rooms to arrange their corresponding relatives, at this time, the boss had no
idea, neither did the boss’s daughter.

(1) If you were the customer manager of Hilbert Hotel (The Grand Hotel), how can
you try to solve the new problem on that third day?

(2) After that, the daughter of the boss went to the university. Some day, Prof. Cantor
asked her, ‘If each point in the range [0, 1] occupies one room, can you arrange
the rooms? ’ Please answer the question raised by Prof. Cantor and show that
with proofs.

(15’, first subproblem 1 worthy 6 marks, subproblem 2 worthy 9 marks, giving
correct proofs earn 6 marks)

Solutions: (1) You can find this in your textbook, the detail description of the Hilbert
Hotel problem. Page 176, exercise 9, we see each person’s relative sub-group a bus,
then the problem’s solution is:

SOLUTION

Since Hilbert’s Grand Hotel has a countably infinite number of rooms, we
can number the rooms by positive integers \( \mathbb{Z}^+ : 1, 2, 3, 4, \ldots \).

We need to spread all guests in a countably infinite number of buses which
contain a countable infinite number of guests. Moreover, the hotel is also full.

Let \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) such that \( f(n) \) represents the \( n \)th bus.
Let \( g_n : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) such that \( g_n(m) \) represents the \( m \)th guest on the \( n \)th bus.

Let us first placed the first guest of each of the buses into rooms in the
building: original guest of room 1 remains in room 1, \( g_1(1) \) is placed in
room 2, \( g_2(1) \) is placed in room 3, \( g_3(1) \) is placed in room 4, ...

Next, we place the second guest of each of the buses into rooms in the
building: original guest of room 2, \( g_2(1), g_2(2), \ldots \) are placed in the next
rooms (while the room number increases by 1 each time a guest is placed in
a room).

Next, we place the third guest of each of the buses into rooms in the building:
original guest of room 3, \( g_3(1), g_3(2), \ldots \) are placed in the next rooms (while
the room number increases by 1 each time a guest is placed in a room).

Repeat this pattern until all passengers have been placed in a room.

We then note that all current guests can be accommodated without evicting
any current guest.

Solution: first, try to understand the problem.

1) $R$ is the equivalence relation on set $A \Rightarrow$ The equivalence classes of $R$ form a partition of set $A$, which is called the quotient set of set $A$ on relation $R$, denoted by $A/R$.

In other words, if $R$ is an equivalence relation on a set $A$, the equivalence class of the element $a$ is $[a]_R = \{m | (a,m) \in R\}$.

2) Choose one fraction of the partition, if there are $k$ elements in the fraction, then the ordered pair number is $k^2$.

3) Sum all the $r$ parts’ ordered pairs, the sum is $s$. The total number of elements of $r$ fractions, the elements number is $n$.

Let the elements number of $r$ equivalence classes, $A/R$ be $m_1, m_2, ..., m_r$ accordingly. Then $m_1+m_2+...+m_r=n$, $m_1^2+m_2^2+...+m_r^2=s$. The original problem changed into the proof of $r(m_1^2+m_2^2+...+m_r^2)/r \geq ((m_1+m_2+...+m_r)/r)^2$, that is $r(m_1^2+m_2^2+...+m_r^2) \geq (m_1+m_2+...+m_r)^2$. (Can be proved using mathematical induction).