Discrete Mathematics

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Chapter Syllabus 课程大纲

Chapter 0 Introduction
Chapter 1 Set Theory
Chapter 2 Counting
Chapter 3 Binary Relation
Chapter 4 Special Relations
Chapter 5 Functions
Chapter 6 Graph Theory
Chapter 7 Trees
Chapter 8 Special Graphs
Goals (章节学习目标):
1. Knowing about the definition of Binary Relation (了解)
2. Knowing about the Operation of Binary Relation （了解）

Lay the foundation of function…
What are relations?

• Relations are a formal means to specify which elements from two or more sets are related to each other

• Examples
  • \{students\} who take \{courses\}
  • \{businesses\} and their \{telephone numbers\}
  • \{integers\} and their \{divisors\}
  • \{program variables\} and the \{subroutines\} they are used in
3 Binary Relation

Definition of relations

- Let $A$ and $B$ be two sets. A *binary relation $R$ from $A$ to $B$* is a subset of $A \times B$.
  - Note that the order of the two sets matters.
- More generally, let $A_1, A_2, ..., A_n$ be $n$ sets. An *$n$-ary relation $R$* on these sets is a subset of $A_1 \times A_2 \times ... \times A_n$.
  - The sets $A_i$ are known as the *domains* of the relation, and $n$ as its *degree*.
  - Again, the order of the domains matters.
The notation \( a R b \) or \( aRb \) means that \((a,b) \in R\).

E.g., \( a < b \) means \((a,b) \in <\)

If \( aRb \) we may say “\( a \) is related to \( b \) (by relation \( R \))”,
or just “\( a \) relates to \( b \) (under relation \( R \))”. 
3 Binary Relation

Domain, Range, Field of Relation R

$$\text{dom}R = \{x \mid \exists y(<x,y> \in R)\}.$$  
$$\text{ran}R = \{y \mid \exists x(<x,y> \in R)\}.$$  
$$\text{fld}R = \text{dom}R \cup \text{ran}R.$$
3 Binary Relation

Cartesian Product

\[ A \times B = \{ \langle x, y \rangle \mid x \in A \land y \in B \}. \]

例如, \( A = \{ a, b \}, B = \{ 0, 1, 2 \} \), 则

\[ A \times B = \{ \langle a, 0 \rangle, \langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 0 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle \}; \]

\[ B \times A = \{ \langle 0, a \rangle, \langle 0, b \rangle, \langle 1, a \rangle, \langle 1, b \rangle, \langle 2, a \rangle, \langle 2, b \rangle \}. \]
Binary Relations

**Definition:** A *binary relation* $R$ from a set $A$ to a set $B$ is a subset $R \subseteq A \times B$.

**Example:**
- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from $A$ to $B$.
- We can represent relations from a set $A$ to a set $B$ graphically or using a table:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>1</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\times$</td>
</tr>
</tbody>
</table>

Relations are more general than functions. A function is a relation where exactly one element of $B$ is related to each element of $A$. 

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Binary Relation on a Set

**Definition:** A binary relation \( R \) on a set \( A \) is a subset of \( A \times A \) or a relation from \( A \) to \( A \).

**Example:**

- Suppose that \( A = \{a, b, c\} \). Then \( R = \{(a, a), (a, b), (a, c)\} \) is a relation on \( A \).
- Let \( A = \{1, 2, 3, 4\} \). The ordered pairs in the relation \( R = \{(a, b) \mid a \text{ divides } b\} \) are \( (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), \) and \( (4, 4) \).
Binary Relation on a Set (cont.)

**Question:** How many relations are there on a set $A$?

**Solution:** Because a relation on $A$ is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has $n^2$ elements when $A$ has $n$ elements, and a set with $m$ elements has $2^m$ subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set $A$. 
Binary Relations on a Set (cont.)

**Example**: Consider these relations on the set of integers:

\[
R_1 = \{(a, b) \mid a \leq b\}, \quad R_4 = \{(a, b) \mid a = b\},
\]

\[
R_2 = \{(a, b) \mid a > b\}, \quad R_5 = \{(a, b) \mid a = b + 1\},
\]

\[
R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}, \quad R_6 = \{(a, b) \mid a + b \leq 3\}.
\]

- Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

\[(1,1), (1, 2), (2, 1), (1, -1), \text{ and } (2, 2)\]?

**Solution**: Checking the conditions that define each relation, we see that the pair \((1,1)\) is in \(R_1, R_3, R_4, \text{ and } R_6\); \((1,2)\) is in \(R_1\) and \(R_6\); \((2,1)\) is in \(R_2, R_5, \text{ and } R_6\); \((1, -1)\) is in \(R_2, R_3, \text{ and } R_6\); \((2,2)\) is in \(R_1, R_3, \text{ and } R_4\).
Powers of a Relation

**Definition:** Let $R$ be a binary relation on $A$. Then the powers $R^n$ of the relation $R$ can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

(see the slides for Section 9.3 for further insights)

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

**Theorem 1:** The relation $R$ on a set $A$ is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

(see the text for a proof via mathematical induction)
Example

- Let $A=\{1,2,3\}$ and $B=\{1,2,3,4\}$. The relations $R_1=\{(1,1),(2,2),(3,3)\}$ and $R_2=\{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain
- $R_1 \cap R_2 = \{(1,1)\}$
- $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$
- $R_1 - R_2 = \{(2,2),(3,3)\}$
- $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$
Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose \( R \) is a relation from \( A = \{a_1, a_2, ..., a_m\} \) to \( B = \{b_1, b_2, ..., b_n\} \).
  - The elements of the two sets can be listed in any particular arbitrary order. When \( A = B \), we use the same ordering.
- The relation \( R \) is represented by the matrix \( M_R = [m_{ij}] \), where
  \[
  m_{ij} = \begin{cases} 
  1 & \text{if } (a_i, b_j) \in R, \\
  0 & \text{if } (a_i, b_j) \notin R.
  \end{cases}
  \]
- The matrix representing \( R \) has a 1 as its \((i,j)\) entry when \( a_i \) is related to \( b_j \) and a 0 if \( a_i \) is not related to \( b_j \).
Examples of Representing Relations Using Matrices

**Example 1:** Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let $R$ be the relation from $A$ to $B$ containing $(a,b)$ if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing $R$ (assuming the ordering of elements is the same as the increasing numerical order)?

**Solution:** Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
Examples of Representing Relations Using Matrices (cont.)

**Example 2:** Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation $R$ represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

**Solution:** Because $R$ consists of those ordered pairs $(a_i, b_j)$ with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$
Representing Relations Using Digraphs

**Definition:** A *directed graph*, or *digraph*, consists of a set $V$ of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called *edges* (or *arcs*). The vertex $a$ is called the *initial vertex* of the edge $(a,b)$, and the vertex $b$ is called the *terminal vertex* of this edge.

- An edge of the form $(a,a)$ is called a *loop*.

**Example 7:** A drawing of the directed graph with vertices $a$, $b$, $c$, and $d$, and edges $(a,b)$, $(a,d)$, $(b,b)$, $(b,d)$, $(c,a)$, $(c,b)$, and $(d,b)$ is shown here.
Examples of Digraphs Representing Relations

**Example 8:** What are the ordered pairs in the relation represented by this directed graph?

**Solution:** The ordered pairs in the relation are 
(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)
Properties of Relations

• Reflexive  自反
• Symmetric  对称
• Transitive  传递
composition

• The composition of two relations $R : A \rightarrow B$ and $S : B \rightarrow C$, denoted $S \circ R$, is the relation from $A$ to $C$ containing all pairs $(x, z)$ such that there is one $y \in B$ with $(x, y) \in R$ and $(y, z) \in S$. 
Homework 1 & 2 Solutions

6. Prove the identity laws in Table 1 by showing that:
   a) \( A \cup \emptyset = A \)
   b) \( A \cap \emptyset = A \)

Solution:
   a) \( A \cup \emptyset = \{x | x \in A \lor x \in \emptyset \} = \{x | x \in A \} = A \) (Set-Builder Notation)
   b) \( A \cap \emptyset = \{x | x \in A \land x \in \emptyset \} = \{x | x \in A \} = A \)

Hints:
   1. Read the definition of propositional variables on page 3 of your textbook.
      Let \( p \) be a propositional variable. The true value of a proposition is true, denoted by \( T \), if it is a true proposition. The negation of \( p \) “\( \neg p \)” is read “not \( p \)”
      Let \( p \) and \( q \) be propositions. The conjunction of \( p \) and \( q \), denoted by \( p \land q \)
      is the proposition “\( p \) and \( q \)” The conjunction \( p \land q \) is true when both \( p \) and \( q \) are true and false otherwise.

   2. Review set builder notations.

9. Prove the idempotent laws in Table 1 by showing that:
   a) \( A \cup A = A \)
   b) \( A \cap A = A \)

Solution:
   a) \( A \cup A = \{x | x \in A \lor x \in A \} = \{x | x \in A \} = A \)
   b) \( A \cap A = \{x | x \in A \land x \in A \} = \{x | x \in A \} = A \)

12. Show that \( A - B = A \cap B' \)

Solution:
   a) \( A - B = \{x | x \in A \land x \in B' \} = \{x | x \in A \land \neg x \in B \} = \{x | x \in A \land \neg x \in B \} = \{x | x \in A \land \neg x \in B \} = A \cap B' \)
   b) \( B - A = \{x | x \in B \land x \in A' \} = \{x | x \in B \land \neg x \in A \} = \{x | x \in B \land \neg x \in A \} = B \cap A' \)

13. Prove the first absorption law from Table 1 by showing that if \( A \) and \( B \) are sets, then \( A \cup (A \cap B) = A \)

Solution:
   a) \( A \cup (A \cap B) \subseteq A \)
   b) \( A \cap (A \cup B) \subseteq A \)

Let \( x \in A \), then \( x \in A \cup (A \cap B) \) so we have proved that \( A \subseteq A \cap (A \cup B) \)
15. Prove the second De Morgan Law in Table 1 by showing that if A and B are sets, then
\[
\overline{A \cup B} = \overline{A} \cap \overline{B}
\]

Hints: Read the logical equivalences on page 27 of your textbook.

1. \(p \land T = p\), \(p \lor F = p\), Identity laws
2. \(p \lor T = T\), \(p \land F = F\), Domination laws
3. \(p \lor p = p\), \(p \land p = p\), Idempotent laws
4. \(\overline{\overline{p}} = p\), Double negation laws
5. \(p \land q = g \land p\), \(p \lor q = g \lor p\), Commutative laws
6. \((p \land q) \lor y = p \lor (q \lor y)\), Associative laws
7. \(p \lor (q \land r) = (p \lor q) \land (p \lor r)\), Distributive laws
8. \(\overline{p} \lor q = \overline{p} \land \overline{q}\) De Morgan's Laws
9. \(\overline{p} \lor \overline{q} = p\), \(p \land \overline{q} = p\), Absorption laws
10. \(p \land \overline{p} = F\), \(p \lor \overline{p} = T\), Negation laws

Solution: \(x \in A \cup B = x \in A \lor x \in B = \overline{\overline{x} \lor \overline{A} \land \overline{B} \land \overline{E}} \overline{T (\overline{X} \land \overline{A}) \land (\overline{X} \land \overline{B})} \overline{\overline{T (\overline{X} \land \overline{A}) \land (\overline{X} \land \overline{B})}}\) for Prop Logic

\(= \overline{x} \lor \overline{A} \land \overline{B} = \overline{x} \lor A \lor B\)
1. Prove the first associative law from Table I by showing that if \( A, B, \) and \( C \) are sets, then

\[ A \cup (B \cup C) = (A \cup B) \cup C \]

Solution: \( x \in A \cup (B \cup C) \) and \( x \in C \) imply \( x \in A \cup B \) and \( x \in A \cup C \), so \( x \in (A \cup B) \cup C \). Conversely, \( x \in (A \cup B) \cup C \) implies \( x \in A \cup (B \cup C) \).

23. \( x \in A \cup (B \cap C) \) implies \( x \in A \cup B \) and \( x \in A \cup C \), and \( x \in B \cap C \) implies \( x \in A \cup B \) and \( x \in A \cup C \), so \( x \in (A \cup B) \cap (A \cup C) \) and \( x \in (A \cup B) \cap (A \cup C) \). Conversely, \( x \in (A \cup B) \cap (A \cup C) \) implies \( x \in A \cup (B \cap C) \).

27. Draw the Venn Diagrams for each of these combinations of the sets \( A, B, C \):

(a) \( A \cap (B \cap C) \)

Solution: \( x \in A \) and \( x \in B \) and \( x \in C \) imply \( x \in A \) and \( x \in B \) and \( x \in C \), so \( x \in A \cap (B \cap C) \). Conversely, \( x \in A \cap (B \cap C) \) implies \( x \in A \) and \( x \in B \) and \( x \in C \).

35. Show that \( A \Delta B = (A \cup B) - (A \cap B) \).

Hints: Review symmetric difference. \( A \Delta B = (A - B) \cup (B - A) \).

Method 1: \( x \in (A \cup B) - (A \cap B) \) implies \( x \in A \cup B \) and \( x \notin A \cap B \), so \( x \in A \) or \( x \in B \), but not both. Conversely, \( x \in A \) and \( x \notin B \) or \( x \in B \) and \( x \notin A \) implies \( x \in A \Delta B \).
40. Determine whether the symmetric difference is associative; that is, if $A$, $B$, and $C$ are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

Hints: In the proof, we use the definition of symmetric difference, the distributive law, DeMorgan’s law and the following fact repeatedly: for any two subsets $X$ and $Y$ of $U$, $X - Y = X \cap \overline{Y}$. You should verify this yourself. Combining the first and last of these we see that $X \oplus Y = (X \cap \overline{Y}) \cup (Y \cap \overline{X})$. 
Homework 2:

14. Let \( n \) and \( r \) be positive integers. Explain why the number of solutions of the equation
\[
X_1 + X_2 + \ldots + X_r = n
\]
where \( X_i \) is a nonnegative integer for \( i = 1, 2, \ldots, r \), equals the number of
\( r \)-combinations of a set with \( n \) elements.

Hints:
1. \( P(n, r) = \frac{n!}{(n-r)!} \)
2. \( C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \)
3. Binomial theorem: \( (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \)
4. Pascal's identity: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \)
5. The number of \( r \)-permutations of a set of \( n \) objects with repetition allowed
is \( n^r \).
6. There are \( \binom{n+r-1}{r} \) \( r \)-combinations from a set with \( n \) elements
when repetition of elements is allowed.

Proof:
Each \( r \)-combination of a set with \( n \) elements when repetition is allowed can be
represented by a list of \( n \) bars and \( r \) stars. The \( n-i \) bars are used to mark off \( n-i \)
different cells, with the \( i \)-th cell containing a star for each time the \( i \)-th element of
the set occurs in the combination. For instance, a \( 3 \)-combination of a set with four-
the set occurs in the combination. For instance, a \( 6 \)-combination of a set with four-
elements is represented with three bars and six stars. Here, XXX | XXX | XXX | XXX represents the
combination containing exactly two of the first element, two of the second element, none of
the third element, and three of the fourth element of the set. As we have seen, each
list containing \( n \) bars and \( r \) stars corresponds to an \( r \)-combination of the set
with \( n \) elements, when repetition is allowed. The number of such lists is \( \binom{n+r-1}{r} \)
because each list corresponds to a choice of the \( r \) positions to place the \( r \) stars from
the \( n-1 \) positions that containing \( n \) stars and \( n \) bars. The number of such
lists is also equal to \( \binom{n+r-1}{n-1} \), because each list corresponds to a choice of the
\( n \) positions to place the \( n \) bars.

b) How many solutions in nonnegative integers are there to the equation
\[
X_1 + X_2 + X_3 + X_4 = 17
\]
(c) How many solutions in positive integers are there to the equation, in particular:
Supplementary Exercises P.441.

15. a) How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces are chosen?

Solution: For worst case, 48 other cards without drawing any of the four aces, so we need 50 cards.

b) a) and at least two of the 13 kinds are chosen? 48 + 2 = 50

c) To guarantee that there are at least two cards of the same kind?

- boxes: 13 different kinds. pigeonhole principle. 13 + 1 = 14

The worst case is if we choose each of the 12 kinds first without repeating any of them. Then the 13th card must be of the same kind as one of the previously chosen cards.

d) To guarantee that there are at least two cards of each of two different kinds?

5 cards of each kind in a standard deck of cards. Thus, if we choose 5 cards then no matter what, at least two of them have to be of different kind.

25. Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with \( A \subseteq B \)?

Solution: Let’s fix B and find the number of satisfying sets A. If \( |B| = k \), then there are \( 2^k \) possible choices for A. Now, let’s sum over all choices of B. For each \( k \), there are \( \binom{n}{k} \) choices for B, each with \( 2^k \) choices for A, giving

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k = (2+1)^n = 3^n
\]
Use distinguishable objects into distinguishable boxes, \( \frac{\text{box}}{\text{i-th}} \).

When the i-th box has to be filled (i = 1, 2, ..., k), the number of options in which the task may be done is same as the no. of ways to choose \( n_i \) objects from the remaining \( n - \sum_{j=1}^{i-1} n_j \), which is \( \left( \frac{n - \sum_{j=1}^{i-1} n_j}{n_i} \right) \). Using product rule:

\[
\prod_{i=1}^{k} \left( \frac{n - \sum_{j=1}^{i-1} n_j}{n_i} \right) = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\]
31. Show that \( \sum_{i=1}^{n} \binom{n}{i} = 2^n \), \( n \geq 3 \).
Solution: Proof: left side = \( \sum_{i=0}^{n} \binom{n}{i} = 2^n \), for \( n \geq 3 \).

35. A professor writes 20 multiple-choice questions, each with the possible answer a, b, c, or d, for a discrete math test. If the number of questions with a, b, c, and d as their answer is 8, 3, 4, and 5, respectively, how many different answer keys are possible if the questions can be placed in any order?
Solution: Distributing 20 distinguishable objects into 4 distinguishable boxes such that no objects are placed in box 1 (a, b, c, or d) can be done in \( \frac{20!}{(1!)(8!)(3!)(4!)(5!)} \) ways.

39. How many solutions are there to the equation \( x_1 + x_2 + x_3 = 17 \), where \( x_1, x_2, x_3 \) are non-negative integers with
a) \( x_1 \geq 1 \), \( x_2 \geq 2 \), \( x_3 \geq 3 \)?
b) \( x_1 < 6 \), \( x_2 > 5 \)?
c) \( x_1 < 4 \), \( x_2 < 5 \), \( x_3 > 7 \)?
Solution: a) Define \( y_1 = x_1 - 1 \), \( y_2 = x_2 - 2 \), \( y_3 = x_3 - 3 \), \( y_1 + y_2 + y_3 = 17 - 3 - 2 - 3 = 10 \).

The number of the solution does not matter. We need to select 8 values from 13 boxes (10 + 3) \( \binom{13}{8} = 1,800 \).

b) Number of solutions with \( x_1 < 6 \), \( x_2 > 5 \).
\( x_1 + x_2 + x_3 = 17 \) \( \binom{3+11}{11} = 7,800 \).

Number of solutions with \( x_1 > 6 \), \( x_2 > 5 \), \( x_3 > 7 \).
\( x_1 + x_2 + x_3 = 17 - 5 - 5 \), \( n = 3 \), \( r = 5 \), \( \binom{3+5}{5} = 21 \).

Number of solutions \( x_1 < 6 \), \( x_2 < 5 \), \( x_3 < 7 \).
\( \binom{17}{3} = 680 \).
Let $n$ and $k$ be integers with $1 \leq k \leq n$. Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n}{k-1}$.

**Proof.** Observe that $\binom{n}{k}$ denotes the number of arrangements in which $n$ people can be seated around $k$ tables so that no table remains empty, $1 \leq k \leq n$.

$\binom{n}{k}$ indicates in how many ways $(n+1)$ people can be seated around $k$ tables which can be done in two ways: Fix a person and let him/her be seated alone at a table. The remaining $n$ people can be seated around the rest $(k-1)$ tables in $\binom{n}{k-1}$ different ways. Next, don't let that same person be seated alone at a table. This means first let the other $n$ people sit around the $k$ tables so that no one of them is empty, in exactly $\binom{n}{k}$ ways, and for each of these ways, we can have our fixed person be seated at any of the $k$ tables, but the problem starts here: there are more than one ways to get that person seated at a table that already contains more than one person. This can be solved as follows:

- Count the number of partitions available for that person and not the number of tables available. If a certain table contains $j$ people, then $j$-1 more of the seating arrangement of other people, our fixed person has exactly $j$ possible options (say to the right of each of the $j$ persons sitting at that table) of being seated at that table. Proceeding like this, in total that person has exactly $\binom{n}{k}$ options of sitting at any table (as the total no. of remaining people is 1) for each of the $\binom{n}{k}$ different seating arrangements of the rest of the persons.
Finish all the written homework and submit online.